

# Projectors separating spectra for $L^2$ on symmetric spaces $\mathrm{GL}(n, \mathbb{C})/\mathrm{GL}(n, \mathbb{R})$

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The Plancherel decomposition of  $L^2$  on a pseudo-Riemannian symmetric space  $\mathrm{GL}(n, \mathbb{C})/\mathrm{GL}(n, \mathbb{R})$  has spectrum of  $[n/2]$  types. We write explicitly orthogonal projectors separating spectrum into uniform pieces.

## 1 Formulas for the projectors

**1.1. Problem.** It is well-known that in various problems of non-commutative harmonic analysis spectra split into pieces of different nature.

In 1977 I. M. Gelfand and S. G. Gindikin [5] formulated a problem about an explicit description of a decomposition of  $L^2$  on a semisimple Lie group  $G$  (and, more generally, on a semi-simple pseudo-Riemannian symmetric space) into a direct sum of representations having uniform spectra. They also gave an answer for  $G = \mathrm{SL}(2, \mathbb{R})$  in the terms of boundary values of holomorphic functions. After this work the question became a topic of intensive discussions and series of works.

The picture is quite nice for  $L^2(\mathrm{SL}(2, \mathbb{R}))$  and  $L^2(\mathrm{SL}(2, \mathbb{R})/H)$ , where  $H$  is the diagonal subgroup, see [7], [8], [1], and [14], [17] respectively. However, up to now the situation for higher groups is far to be well-understood.

**1.2. Known approaches.** a) G. I. Olshanski [21] proposed a way to split off holomorphic discrete series using boundary values of holomorphic functions, this approach was used in several works, see, e.g. [13], [12].

S. G. Gindikin, B. Krötz and G. Ólafsson [9] showed that in some cases an integral of the most continuous series can be splitted off in a similar way.

b) V. F. Molchanov [15] and S. G. Gindikin [8] in different way obtained the desired decomposition for  $L^2$  on multi-dimensional hyperboloids  $\mathrm{O}(p, q)/\mathrm{O}(p, q-1)$  (this includes the cases  $L^2(\mathrm{SL}(2, \mathbb{R}))$  and  $L^2(\mathrm{SL}(2, \mathbb{R})/H)$  mentioned above).

c) In [16] there was proposed a way for separation of summands of complementary series, see more in [18], [20].

**1.3. Purposes of the present work.** For a pseudo-Riemannian symmetric space  $\mathrm{GL}(n, \mathbb{C})/\mathrm{GL}(n, \mathbb{R})$  consider a decomposition of  $L^2$  into a direct sum of subspaces, in which  $\mathrm{GL}(n, \mathbb{C})$  has uniform spectra<sup>2</sup>,

$$L^2(\mathrm{GL}(n, \mathbb{C})/\mathrm{GL}(n, \mathbb{R})) = L_0 \oplus L_1 \oplus \cdots \oplus L_{[n/2]}. \quad (1.1)$$

We also have a corresponding decomposition of the identity operator:

$$E = \Pi_0 \oplus \Pi_1 \oplus \cdots \oplus \Pi_{[n/2]}, \quad (1.2)$$

<sup>1</sup>Supported by the grants FWF, P25142, P28421

<sup>2</sup>Here  $[n/2]$  denotes the integer part of  $n/2$ .

where  $\Pi_r$  is the orthogonal projector to a subspace  $L_r$ . We intend to obtain explicit expressions for the projectors  $\Pi_r$ .

The present work is based on both results and auxiliary calculations of Shigeru Sano [22]. The formulas for the projectors are given by Theorem 2.1.

#### 1.4. The spaces $\mathrm{GL}(n, \mathbb{C})/\mathrm{GL}(n, \mathbb{R})$ . Below

$$G := \mathrm{GL}(n, \mathbb{C}), \quad H := \mathrm{GL}(n, \mathbb{R}).$$

We realize the symmetric space  $G/H$  as the space  $M$  of all matrices  $x \in \mathrm{GL}(n, \mathbb{C})$  satisfying the condition

$$x\bar{x} = 1.$$

The group  $G$  acts on  $M$  by transformations

$$g : x \mapsto x \circ g := g^{-1}x\bar{g},$$

the stabilizer of  $x = e$  is  $H$ . Notice that  $H$  acts on  $M$  by conjugations.

Denote by  $C_c^\infty(G/H)$  the space of  $C^\infty$ -smooth functions on  $G/H$  with compact support. Let  $\chi$  be an  $H$ -invariant distribution on  $G/H$ . It determines a  $G$ -intertwining operator

$$A[\chi] : C_c^\infty(G/H) \rightarrow C^\infty(G/H)$$

by the pairing<sup>3</sup>

$$A[\chi]f(g) = \langle f(x \circ g, \chi) \rangle_{G/H},$$

a function on  $G$  obtained in this way is  $H$ -invariant and therefore it is a function on  $G/H$ .

Our purpose is to write  $H$ -invariant distributions on  $G/H$  determining the projectors  $\Pi_r$ .

**1.5. Notation.** By  $dx$  we denote a  $G$ -invariant measure on  $G/H$  (it is unique up to a constant factor). Denote by  $S_m$  the symmetric group of order  $m$ , by  $\mathbb{T}^p$  the torus  $(\mathbb{R}/2\pi\mathbb{Z})^p$ . Denote by  $\Delta$  the *Vandermonde expression*:

$$\Delta(s) = \Delta(s_1, \dots, s_n) := \prod_{1 \leq p < q \leq n} (s_p - s_q). \quad (1.3)$$

**1.6. Cartan subspaces and the Weyl integration formula.** Let  $\varphi$ ,  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ ,  $t \in \mathbb{R}$ . Denote by  $v(\theta, t)$  a  $2 \times 2$  matrix given by

$$v(\theta, t) = e^{i\theta} \begin{pmatrix} \cosh t & i \sinh t \\ -i \sinh t & \cosh t \end{pmatrix} = \begin{pmatrix} e^{i\theta} \cosh t & ie^{i\theta} \sinh t \\ -ie^{i\theta} \sinh t & e^{i\theta} \cosh t \end{pmatrix},$$

its eigenvalues are

$$e^z := e^{t+i\theta}, \quad e^{-\bar{z}} := e^{-t+i\theta}.$$

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<sup>3</sup>By  $\langle f, \chi \rangle_L$  we denote a pairing of a test function  $f$  and a distribution  $\chi$  on a manifold  $L$ .

We define a *Cartan subspace*  $A_k$ , where  $k = 1, 2, \dots, [n/2]$ , as the set of matrices  $a \in M$  having the following block-diagonal form

$$a^{(k)} := \begin{pmatrix} e^{i\varphi_1} & & & & \\ & \ddots & & & \\ & & e^{i\varphi_{n-2k}} & & \\ & & & v(\theta_k, t_k) & \\ & & & & \ddots \\ & & & & & v(\theta_1, t_1) \end{pmatrix} \quad (1.4)$$

We equip  $A_k$  with the standard Lebesgue measure

$$da^{(k)} := \prod_{l=1}^{n-2k} d\varphi_l \prod_{m=1}^k (d\theta_m dt_m).$$

The relative *Weyl group*  $W_k$  corresponding to the Cartan subspace  $A_k$  is

$$W_k \simeq S_{n-2k} \times (S_k \ltimes \mathbb{Z}_2^k).$$

The factor  $S_{n-2k}$  acts on  $A_k$  by permutations of  $e^{i\varphi_2}, \dots, e^{i\varphi_{n-2k}}$ , the group  $S_k$  acts by permutations of pairs  $(\theta_1, t_1), \dots, (\theta_k, t_k)$ . The group  $\mathbb{Z}_2^k$  is generated by reflections

$$R_m : (t_1, \dots, t_{m-1}, t_m, t_{m+1}, \dots, t_k) \mapsto (t_1, \dots, t_{m-1}, -t_m, t_{m+1}, \dots, t_k) \quad (1.5)$$

(all the coordinates  $\varphi$  and  $\theta$  remain to be fixed). We denote

$$\gamma_k := \frac{1}{\#W_k} = \frac{1}{k!(n-2k)!2^k}. \quad (1.6)$$

**1.7. Averaging operators.** An element of  $M$  having different eigenvalues can be reduced to one of subalgebras  $A_k$  by a conjugation by some  $h \in H$ . This element  $a \in A_k$  is defined up to the action of  $W_k$ .

Define a function  $\Delta_k$  on  $A_k$  by the Vandermonde expression

$$\Delta_k(a^{(k)}) = \Delta(e^{i\varphi_1}, \dots, e^{i\varphi_{n-2k}}, e^{z_k}, e^{-\bar{z}_k}, \dots, e^{z_1}, e^{-\bar{z}_1}).$$

Notice that  $\Delta_k(a^{(k)})$  are pure real or pure imaginary depending on  $n, k$ ; also the sign of  $\Delta_k(a^{(k)})$  depends on the choose of order of eigenvalues.

On the other hand, denote by  $B_k \subset H$  the subgroup fixing all elements of  $A_k$ . It consists of block diagonal real matrices with  $n-2k$  blocks of size  $1 \times 1$  and  $k$  blocks of size  $2 \times 2$  having the form  $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ ,  $c^2 + d^2 \neq 0$ . Consider a map  $A_k \times (H/B_k) \rightarrow M$  given by

$$(a^{(k)}, y) \mapsto ya^{(k)}y^{-1}. \quad (1.7)$$

An element  $y \in H$  is a matrix determined up to an equivalence  $y \sim yb$ , where  $b \in B_k$ . Therefore the map is well-defined. Equip  $H/B_k$  with an invariant

measure  $d_k y$ , which will be normalized several lines below. For any function  $f \in C_c^\infty(G)$  we define a collection of functions  $I_k f$  on  $A_k$  (integrals of  $f$  over orbits of  $H$ ):

$$I_k f(a^{(k)}) = \int_{H/B_k} f(ya^{(k)}y^{-1}) d_k y$$

(functions  $I_k f$  are well-defined on the set  $\Delta_k(a^{(k)}) \neq 0$ ). Then under a certain normalization of measures  $dx$  and  $d_k y$  we have the identity

$$\int_{G/H} f(x) dx = \sum_{k=0}^{[n/2]} \gamma_k \int_{A_k} I_k f(a^{(k)}) |\Delta(a^{(k)})|^2 da^{(k)}. \quad (1.8)$$

This is a version of the Weyl integration formula. At each point  $a^{(k)} \in A_k$  the volume form on  $G/H$  is a product of forms

$$|\Delta(a^{(k)})|^2 da^{(k)} \wedge d_k y$$

and this explains a normalization of  $d_k y$ . The factors  $\gamma_k = 1/\#W_k$  arise because the map (1.7) covers each point of its image  $\#W_k$  times. See [22], Sect.4.

**1.8. Normalized averaging operators.** For each  $k$  we define the expression  $\varepsilon_k(a^{(k)})$  on  $A_k$  by

$$\varepsilon_k(a^{(k)}) := \prod_{1 \leq p < q \leq n-2k} \text{sign} \sin\left(\frac{\varphi_p - \varphi_q}{2}\right).$$

For each  $k$  we define an *averaging operator* from  $C_c^\infty(G)$  to the space of functions on  $A_k$  by

$$\Xi_k f(a^{(k)}) := \varepsilon_k(a^{(k)}) (\det a^{(k)})^{-(n-1)/2} \overline{\Delta_k(a^{(k)})} \int_{H/B_k} f(ya^{(k)}y^{-1}) d_k y. \quad (1.9)$$

The functions  $\Xi_k f$  are invariant with respect to the action of the subgroups  $S_k, S_{n-2k} \subset W_k$  and change signs under the reflections  $R_m$ , see (1.5). The hypersurfaces  $t_m = 0$  divide  $A_k$  into a union of  $2^k$  'octants'

$$\left\{ t_1 \leq 0, \quad \dots \quad t_k \leq 0. \right. \quad (1.10)$$

Generally,  $\Xi_k f(a^{(k)})$  is discontinuous on these hypersurfaces. A function  $\Xi_k f(a^{(k)})$  admits a smooth continuation to the closure of each 'octant' and the operator  $\Xi_k$  is a bounded operator in a natural sense ([25], Corollary 8.5.1.2).

An intersection  $A_k \cap A_{k+1}$  of Cartan subspaces is a hypersurface in each subspace and  $\Xi_k f, \Xi_{k+1} f$  satisfy some boundary conditions on this hypersurface (see Subsect. 3.2).

Thus we get an operator  $\Xi$ , which sends a function  $f \in C_c^\infty(G/H)$  to a collection of functions

$$\Xi : f \mapsto (\Xi_0 f, \dots, \Xi_{[n/2]} f).$$

An  $H$ -invariant distribution on  $G/H$  determines a linear functional on the image of  $\Xi$ .

**1.9. Differential Vandermonde.** Consider the following  $n$ -tuple of first order differential operators on  $A_k$ :

$$\frac{\partial}{i\partial\varphi_1}, \dots, \frac{\partial}{i\partial\varphi_{n-2k}}, \frac{1}{2}\left(\frac{\partial}{\partial t_k} + \frac{\partial}{i\partial\theta_k}\right), \frac{1}{2}\left(-\frac{\partial}{\partial t_k} + \frac{\partial}{i\partial\theta_k}\right), \dots, \frac{1}{2}\left(\frac{\partial}{\partial t_2} + \frac{\partial}{i\partial\theta_2}\right), \frac{1}{2}\left(-\frac{\partial}{\partial t_1} + \frac{\partial}{i\partial\theta_1}\right). \quad (1.11)$$

Denote them by  $X_1, \dots, X_n$ . We define the differential operator

$$\Delta(\partial) = \Delta_k(\partial) := \Delta(X_1, \dots, X_n).$$

Any function  $F = \Delta_k(\partial) \Xi_k f(a^{(k)})$  on  $A_k$  satisfies the following properties (see [22], (8.2)):

$A^\circ$ .  $F$  is skew-symmetric with respect to the subgroup  $S_{n-2k} \subset W_k$  and invariant with respect to of  $S_k \times \mathbb{Z}_2^k$ .

$B^\circ$ .  $F$  admits a continuous extension to the whole  $A_k$ ;

$C^\circ$ .  $F$  is smooth on the closure of each 'octant' (1.10).

There is a constant  $\gamma_*$  depending on a normalization of the measure on  $G/H$  such that

$$\Delta_{[n/2]} \Xi_{[n/2]} f(0) = \gamma_* f(e) \quad (1.12)$$

for any  $f \in C_c^\infty(G/H)$ .

**1.10. Distributions  $\Lambda_p$ .** Recall that a function  $\cot \varphi/2$  determines a distribution on the circle  $\mathbb{R}/2\pi\mathbb{Z}$  by the formula

$$f \mapsto \text{p.v.} \int_{-\pi}^{\pi} \cot \frac{\varphi}{2} f(\varphi) d\varphi.$$

Consider an even-dimensional torus  $\mathbb{T}^{2m}$  with standard coordinates  $e^{i\varphi_1}, \dots, e^{i\varphi_{2m}}$ . Define the distribution  $\Lambda_{2m}$  by

$$\Lambda_{2m}(\varphi) := \frac{(-i)^m}{(2\pi)^m 2^m m!} \sum_{\sigma \in S_{2m}} (-1)^\sigma \prod_{j=1}^m \left( \cot \frac{\varphi_{\sigma(2j-1)}}{2} \cdot \delta(\varphi_{\sigma(2j-1)} + \varphi_{\sigma(2j)}) \right). \quad (1.13)$$

For an odd-dimensional torus  $\mathbb{T}^{2m+1}$  we define the distribution  $\Lambda_{2m+1}$  by

$$\begin{aligned} \Lambda_{2m+1}(\varphi) := & \frac{(-i)^m}{(2\pi)^m 2^m m!} \sum_{\sigma \in S_{2m+1}} (-1)^\sigma \times \\ & \times \delta(\varphi_{\sigma(2m+1)}) \cdot \prod_{j=1}^m \left( \cot \frac{\varphi_{\sigma(2j-1)}}{2} \cdot \delta(\varphi_{\sigma(2j-1)} + \varphi_{\sigma(2j)}) \right). \end{aligned} \quad (1.14)$$

**1.11. Formulas for projectors.** The purpose of the present paper is the following formula.

**Theorem 1.1** *Invariant distributions  $\Theta_r : C_c^\infty(G/H) \rightarrow \mathbb{C}$  determining the projectors  $\Pi_0, \dots, \Pi_{[n/2]}$  in (1.1)–(1.2) are given by the formula*

$$\begin{aligned} \gamma_* \langle f, \Theta_r \rangle_G &= \frac{(-1)^{n(n-1)/2} (n-2r)!}{([n/2]-r)!} \sum_{k=0}^r (-1)^k 4^{r-k} \times \\ &\times \gamma_k \left\langle \Delta_k(\partial) \Xi_k f(\varphi, \theta, t), \Lambda_{n-2k}(\varphi) \cdot \prod_{j=1}^k \delta(t_j) \delta(\theta_j) \right\rangle_{A_k}. \end{aligned} \quad (1.15)$$

In particular, the projector corresponding to the discrete series is determined by the distribution

$$\gamma_* \Theta_0(f) = \frac{(-1)^{n(n-1)/2} (n-2r)!}{[n/2]!} \langle \Delta_0 \Xi_0 f, \Lambda_n \rangle_{A_0}.$$

**1.12. Remarks on the general problem for pseudo-Riemannian symmetric spaces.** The problem of separation of spectra is reduced to integration of spherical distributions as functions of parameters with respect to the Plancherel measure.

The spaces  $GL(n, \mathbb{C})/GL(n, \mathbb{R})$  are representatives of spaces  $G(\mathbb{C})/G(\mathbb{R})$ , where  $G(\mathbb{C})$  is a complex semisimple (reductive) Lie group and  $G(\mathbb{R})$  is its real form. The Plancherel formula for such spaces was obtained by Sh. Sano, N. Bopp, and P. Harink [23], [3], [10]. The spaces  $G(\mathbb{C})/G(\mathbb{R})$  are close relatives and spherical distributions admit elementary expressions similar to (3.10)–(3.11).

In a recent preprint [19] the similar problem was solved for  $L^2$  on the pseudo-unitary group  $U(p, q)$  (formulas for projectors have another form but calculations are similar). In the case of real semisimple groups spherical distributions are characters, according Harish-Chandra characters are locally integrable functions admitting elementary expressions.

For general pseudo-Riemannian semisimple symmetric spaces the way used here and in [19] is impossible. On the other hand, calculations of V. F. Molchanov in rank one case [15] indicate that a general formulation of the problem requires an improvement.

**1.13. Structure of the paper.** The proof of Theorem 1.1 is based on the Plancherel formula obtained by Sano [22] and also on his proof. For this reason, we must expose numerous elements of the paper [22]. In the Section 2 we establish a skew-symmetric counterpart of the formula

$$\sum_{n=-\infty}^{\infty} e^{in\varphi} = 2\pi\delta(\varphi).$$

Section 3 contains preliminaries and Section 4 evaluation of the projectors.

**Acknowledgements.** I am grateful to H. Upmeyer for discussions of this topic.



Figure 1: A matching. Even case and odd case.

## 2 A skew-symmetric analog of the delta-function

**2.1. Distributions  $\Lambda$ .** For integers  $a_1 > \dots > a_p$  denote

$$\mathcal{E}_p^a = \mathcal{E}_p^{a_1, \dots, a_p}(e^{i\varphi_1}, \dots, e^{i\varphi_p}) := \sum_{\sigma \in S_p} (-1)^\sigma e^{i \sum_m a_{\sigma(m)} \varphi_m}. \quad (2.1)$$

Denote by  $\Lambda_p$  the following distribution on a torus  $\mathbb{T}^p$ :

$$\mathcal{L}_p(e^{i\varphi_1}, \dots, e^{i\varphi_p}) := \frac{1}{(2\pi)^p} \sum_{a_1 > \dots > a_p} \mathcal{E}_p^a. \quad (2.2)$$

**Theorem 2.1** *We have*

$$\mathcal{L}_p = \Lambda_p,$$

where  $\Lambda_p$  is given by (1.13), (1.14).

For a proof of this statement we need some combinatorial lemmas.

**2.2. Matchings.** Consider a  $p$ -element ordered set  $C$ , it is convenient to assume  $C = \{p, p-1, \dots, 1\}$ . If  $p$  is even, we say that a *matching* of  $C$  is a partition

$$\zeta : C = \{c_1, c_2\} \sqcup \{c_3, c_4\} \sqcup \dots \quad (2.3)$$

of  $C$  into two-element subsets. If  $p$  is odd, then a *matching* is a partition of  $C$  into  $(p-1)/2$  two-element subsets and one single point subset,

$$\zeta : C = \{c_1, c_2\} \sqcup \dots \sqcup \{c_{p-2}, c_{p-1}\} \sqcup \{c_p\} \quad (2.4)$$

We draw matchings as diagrams with arcs, see Figure 1. Denote by

$$\text{Match}(C) = \text{Match}_p$$

the set of all matchings. Define the standard matching  $\zeta_0$  by

$$\zeta_0 = \begin{cases} \{p, p-1\} \sqcup \{p-2, p-3\} \sqcup \dots \sqcup \{2, 1\}, & \text{if } p \text{ is even;} \\ \{p, p-1\} \sqcup \{p-2, p-3\} \sqcup \dots \sqcup \{3, 2\} \sqcup \{1\}, & \text{if } p \text{ is odd.} \end{cases}$$

The symmetric group  $S_p$  acts on  $\text{Match}_p$  in a natural way. It is convenient to imagine this action as on Figure 2 (we glue a diagram of matching with a diagram of a permutation).

Consider a matching (2.3) or (2.4). We say that pairs  $\{c_\alpha, c_{\alpha+1}\}$  and  $\{c_\beta, c_{\beta+1}\}$  are *interlacing* if precisely one point  $c_\beta$  or  $c_{\beta+1}$  lies between  $c_\alpha$  and  $c_{\alpha+1}$  (in the sense of the ordering of  $C$ ). On Figure 1 an interlacing of pairs corresponds to

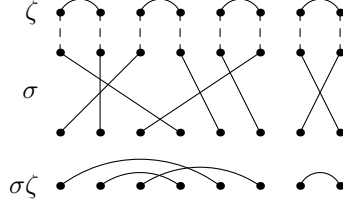


Figure 2: Action of symmetric group on  $\text{Match}_p$ .

an intersection of arcs. If  $\#C = p$  is odd, we say that a distinguished element  $\{c_p\}$  of a matching *interlaces* a pair  $\{c_\alpha, c_{\alpha+1}\}$  if  $c_p$  lies between  $c_\alpha$  and  $c_{\alpha+1}$ .

We say that a matching  $\zeta$  is *even* (respectively, *odd*) if the number of pairs of interlacing elements of  $\zeta$  is even (respectively, odd). Denote the parity by  $(-1)^\zeta$ .

**Lemma 2.2** *For any  $p$ ,*

$$\Sigma(p) := \sum_{\zeta \in \text{Match}_p} (-1)^\zeta = 1. \quad (2.5)$$

PROOF. First, let  $p$  be even. Define an involution  $J$  of  $\text{Match}_p$  in the following way.

— If  $\{p, p-1\}$  is not an element of a matching  $\zeta$ , we apply to  $\zeta$  the transposition  $(p, p-1) \in S_p$ .

— If  $\{p, p-1\}$  is an element of  $\zeta$  and  $\{p-2, p-3\}$  not, we apply to  $\zeta$  the transposition  $(p-2, p-3) \in S_p$ . Etc.

—  $J$  fixes the matching  $\zeta_0$ .

The involution  $J$  changes parity of all matchings  $\zeta \neq \zeta_0$ . This implies our statement.

Next, let  $p$  be odd. Forgetting a singleton  $\{c_p\}$  in  $\zeta \in \text{Match}_p$ , we get a map  $h : \text{Match}_p \rightarrow \text{Match}_{p-1}$ . Moreover,

$$(-1)^\zeta = (-1)^{c_p+1} (-1)^{h(\zeta)}.$$

For  $c_p = 1, 3, 5, \dots, p$ , we have  $(-1)^\zeta = (-1)^{h(\zeta)}$ , for  $c_p = 2, 4, \dots, 2p-1$ , the signs are different. Therefore  $\Sigma(p) = \Sigma(p-1)$ .  $\square$

**Lemma 2.3** *For any permutation  $\sigma \in S_p$*

$$(-1)^{\sigma\zeta_0} = (-1)^\sigma \cdot \text{sign}(\sigma(p) - \sigma(p-1)) \cdot \text{sign}(\sigma(p-2) - \sigma(p-3)) \dots$$

PROOF. We imagine  $\sigma \in S_p$  as a bipartite graph as on Figure 2. Inversions in permutations correspond to intersections of arcs. Parity of a permutation is parity of number of intersections. A matching  $\sigma\zeta_0$  is obtained by gluing the



matching  $\zeta_0$  and the permutation  $\sigma$ . Multiplication of  $(-1)^\sigma$  by  $\prod_{j=0}^{p-1} \text{sign}(\sigma(p-2j) - \sigma(p-2j-1))$  means that we do not take to account possible inversions in pairs  $(p-2j, p-2j-1)$ . Now the statement must be clear from Figure 2.  $\square$

**2.3. A transformation of the expression for  $\Lambda_p$ .** Now we can write formulas (1.13), (1.14) in the form

$$\Lambda_p(e^{i\varphi_1}, \dots, e^{i\varphi_p}) = (-i)^{[p/2]} \sum_{\zeta \in \text{Match}_p} (-1)^\zeta \prod_{\{\alpha, \beta\} \in \zeta} \cot(\varphi_\alpha/2) (\delta(\varphi_\alpha) + \delta(\varphi_\beta)) \times \prod_{\{\gamma\} \in \zeta} \delta(\varphi_\gamma). \quad (2.6)$$

The product  $\prod_{\{\gamma\} \in \zeta}$  is taken over the set consisting of 0 or 1 elements, i.e., for even  $p$  the product equals 1 and for odd  $p$  it consists of 1 factor.

#### 2.4. Proof of Theorem 2.1.

**Lemma 2.4** *Let  $a_1 > \dots > a_p$ , and  $\mathcal{E}_p^a$  be given by (3.9).*

a)  $p = 2m$  be even. Then

$$i^m \left\langle \mathcal{E}_p^a, \prod_{l=1}^m \left( \cot \frac{\varphi_{2l-1}}{2} \cdot \delta(\varphi_{2l-1} + \varphi_{2l}) \right) \right\rangle_{\mathbb{T}^{2m}} = (2\pi)^{2m} 2^m m!. \quad (2.7)$$

b)  $p = 2m + 1$  be odd. Then

$$i^m \left\langle \mathcal{E}_p^a, \delta(\varphi_{2m+1}) \prod_{l=1}^m \left( \cot \frac{\varphi_{2l-1}}{2} \cdot \delta(\varphi_{2l-1} + \varphi_{2l}) \right) \right\rangle_{\mathbb{T}^{2m}} = (2\pi)^{2m+1} 2^m m!. \quad (2.8)$$

PROOF. We prove a), a proof of b) is same. Recall that

$$i \cot \varphi/2 = \sum_{n>0} (e^{-in\varphi} - e^{in\varphi}).$$

Therefore

$$i^m \left\langle (-1)^\sigma e^{ia_{\sigma(1)}\varphi_1 + ia_{\sigma(2)}\varphi_2 + \dots}, \prod_{l=1}^m \left( \cot \frac{\varphi_{2l-1}}{2} \cdot \delta(\varphi_{2l-1} + \varphi_{2l}) \right) \right\rangle = \quad (2.9)$$

$$\begin{aligned} &= i^m (2\pi)^m (-1)^\sigma \left\langle e^{i(a_1 - a_2)\varphi_1 + i(a_3 - a_4)\varphi_3 + \dots}, \prod_{l=1}^m \cot \frac{\varphi_{2l-1}}{2} \right\rangle = \\ &= (2\pi)^{2m} (-1)^\sigma \prod_{l=1}^m \text{sign}(a_{\sigma(2l-1)} - a_{\sigma(2l)}) = (-1)^{\sigma \zeta_0} (2\pi)^{2m}. \end{aligned} \quad (2.10)$$

Thus (2.10) is

$$(2\pi)^{2m} \sum_{\sigma \in S_p} (-1)^{\sigma \zeta_0} = 2^m m! (2\pi)^{2m} \sum_{\zeta \in \text{Match}_p} (-1)^\zeta.$$

The latter sum was evaluated in Lemma 2.3.  $\square$

To be definite, assume that  $p$  is even. Since  $\mathcal{E}_p^a$  is skew-symmetric, replacing

$$I(\varphi) := \prod \cot \frac{\varphi_{2l-1}}{2} \cdot \delta(\varphi_{2l-1} + \varphi_{2l})$$

in (2.7) by its skew-symmetrization

$$J := \sum_{\sigma \in S_p} (-1)^\sigma I(\varphi_{\sigma(1)}, \dots, \varphi_{\sigma(p)})$$

we get the same right-hand side multiplied by  $p!$ . Next, we replace  $\mathcal{E}_p^a$  by  $\exp(\sum i a_k \varphi_k)$ , after this the right hand side is divided by  $p!$ . Thus we get Fourier coefficients of the complex conjugate distribution  $\overline{J}$ . This is Theorem 2.1.

### 3 The Plancherel formula

This section contains preliminaries from [22].

**3.1. Choose of a sign in the average operator.** For even  $n$  the right hand side of formula (1.9) is defined up to a sign. Let us fix it. Denote

$$\nu_k(a^{(k)}) = \prod_{1 \leq p < q \leq n-2k} \sin\left(\frac{\varphi_p - \varphi_q}{2}\right).$$

We have

$$\begin{aligned} \nu_k(a^{(k)}) (\det a^{(k)})^{-(n-1)/2} &= \prod_{1 \leq p < q \leq n-2k} (2i)^{-1} e^{-i(\varphi_p - \varphi_q)/2} (e^{i(\varphi_p - \varphi_q)} - 1) \times \\ &\quad \times \prod_{p=1}^{n-2k} e^{i\varphi_p(n-1)/2} \cdot \prod_{r=1}^k e^{i\theta_r(n-1)}. \end{aligned}$$

Collecting factors  $e^{i\varphi_m}$  we get a continuous expression

$$e^{-i(\varphi_2 + 2\varphi_3 + \dots + (n-2k-1)\varphi_{n-2k})} \prod e^{i\theta_r(n-1)} \prod (e^{i(\varphi_p - \varphi_q)} - 1),$$

which is well-defined up to global change of a sign. We fix its sign assuming that  $\nu_k(a^{(k)})$  is positive if the order  $e^{i\varphi_1}, \dots, e^{i\varphi_{n-2k}}$  are located clock-wise. It remains to set

$$\varepsilon_k(a^{(k)}) (\det a^{(k)})^{-(n-1)/2} := \frac{\nu_k(a^{(k)}) (\det a^{(k)})^{-(n-1)/2}}{|\nu_k(a^{(k)})|}.$$

**3.2. Conditions of glueing.** Consider a Cartan subspace  $A_k$  (see (1.4) with the standard coordinates

$$\varphi_1, \dots, \varphi_{n-2k-2}, \varphi_{n-2k-1}, \varphi_{n-2k}, t_k, \theta_k, \dots, t_1, \theta_1 \quad (3.1)$$

and the Cartan subspace  $A_{k+1}$  with coordinates

$$\varphi_1, \dots, \varphi_{n-2k-2}, t_{k+1}, \theta_{k+1}, t_k, \theta_k, \dots, t_1, \theta_1. \quad (3.2)$$

The intersection  $A_k \cap A_{k+1}$  in  $A_{k+1}$  is given by the equation  $t_{k+1} = 0$ ; in  $A_k$  it is defined by the equation  $\varphi_{n-2k-1} = \varphi_{n-2k}$ . Let us change coordinates in  $A_k$  and in  $A_{k+1}$  in the following way. In the both cases we leave coordinates

$$\varphi_1, \dots, \varphi_{n-2k-3}, t_k, \theta_k, \dots, t_1, \theta_1. \quad (3.3)$$

being the same. Also:

- in  $A_{k+1}$ , we rename  $t := t_{k+1}$ ,  $\theta := \theta_{k+1}$ ;
- in  $A_k$  we set  $\theta := (\varphi_{n-2k-1} + \varphi_{n-2k})/2$ ,  $\tau := \varphi_{n-2k-1} - \varphi_{n-2k}$ .

Take a point  $b$  of the hypersurface  $t = 0$  such that other  $t_j \neq 0$ . Then for any  $N > 0$  there are smooth functions  $u_j(\cdot)$  depending on coordinates (3.3) and  $\theta$  such that in a small neighborhood of  $b$  we have asymptotic expansions of the form

$$\begin{aligned} \Xi_{k+1}f(t, \dots) &= \sum_{j=0}^N u_j t^j + o(t^N), & \text{for } t > 0; \\ \Xi_{k+1}f(t, \dots) &= \sum_{j=0}^N (-1)^{j+1} u_j t^j + o(t^N), & \text{for } t < 0; \\ \Xi_k f(\tau, \dots) &= i \sum_{m=0}^{[N/2]} (-1)^m u_{2m} \tau^{2m} + o(\tau^N), \end{aligned}$$

see [22], Theorem 4.4.

REMARK. For  $t \geq 0$  denote

$$\Xi_{k+1}^+ f(t, \dots) = (\Xi_{k+1} f(t, \dots) + \Xi_{k+1} f(-t, \dots))/2$$

Then the function

$$\tilde{\Xi}f(s) := \begin{cases} \Xi_{k+1}^+ f(\sqrt{s}, \dots), & \text{for } s \geq 0; \\ i \Xi_k f(\sqrt{-s}, \dots), & \text{for } s \leq 0 \end{cases}$$

is a smooth function near  $s = 0$ . Appearance of the factor  $i$  is artificial, it is related to the normalization of the factor  $\Delta(a^{(k)})$  in (1.9). See also [24] for elementary explanations, our asymptotics can be reduced to the case  $p = 1$ ,  $q = 2$  of that paper.  $\square$

**3.3. Spherical distributions**, see [22], Sect. 7. Points of the spectrum of  $L^2(G/H)$  are enumerated by an integer  $r = 0, \dots, [n/2]$  and signatures of type  $r$ :

$$(c, l) = (c_1, \dots, c_{n-2r}, l_1, \bar{l}_1, \dots, l_r, \bar{l}_r). \quad (3.4)$$

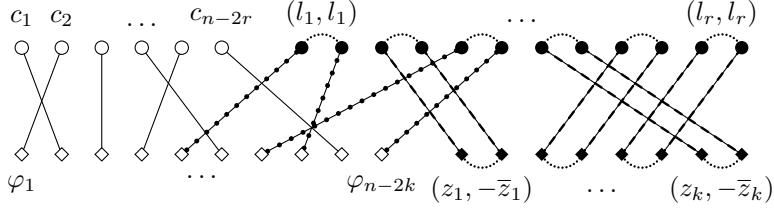


Figure 3: A diagram  $\mathfrak{S}$ .

Here  $c_1 > c_2 > \dots > c_{n-2r}$  are integers, and

$$l_p = (m_p - i\lambda_p)/2,$$

where  $m_p \in \mathbb{Z}$ ,  $\lambda_p \in \mathbb{R}$ , and  $\lambda_1 > \lambda_2 > \dots > \lambda_r > 0$ . To write a distribution corresponding to a given signature  $(c, l)$ , we need some notation.

Let  $\varphi_1, \varphi_2$  be defined modulo  $2\pi$ . Then  $(\varphi_1 - \varphi_2)/2$  is defined modulo  $\pi$ . We will use two ways to define  $(\varphi_1 - \varphi_2)/2$  modulo  $2\pi$  setting

$$\left\lfloor \frac{\varphi_1 - \varphi_2}{2} \right\rfloor \in (-\pi/2, \pi/2), \quad \left\lceil \frac{\varphi_1 - \varphi_2}{2} \right\rceil \in (0, \pi). \quad (3.5)$$

Next, for  $l = (m - i\lambda)/2$  we define a function

$$\xi(l; e^z) := e^{im\theta} (e^{i\lambda t} - e^{-i\lambda t}), \quad \text{where } e^z = e^{t+i\theta}, \quad (3.6)$$

and a function  $D(l; e^{i\varphi_1}, e^{i\varphi_2})$ . If  $m \in 2\mathbb{Z}$ , we set

$$D(l; e^{i\varphi_1}, e^{i\varphi_2}) := 2e^{im(\varphi_1 + \varphi_2)/2} \frac{\cosh \lambda (|\lfloor (\varphi_1 - \varphi_2)/2 \rfloor - \pi/2|)}{\sinh \pi \lambda / 2}. \quad (3.7)$$

If  $m \in 2\mathbb{Z} + 1$ , then (formula (8.6) in [22] contains a typos).

$$D(l; e^{i\varphi_1}, e^{i\varphi_2}) := -2e^{i(m-1)(\varphi_1 + \varphi_2)/2} \times \\ \times e^{i\varphi_2 + \lceil (\varphi_1 - \varphi_2)/2 \rceil} \frac{\sinh \lambda (\lceil (\varphi_1 - \varphi_2)/2 \rceil - \pi/2)}{\cosh \lambda \pi / 2}. \quad (3.8)$$

Functions  $D_l$  are continuous on the torus  $\mathbb{T}^2$ , smooth outside the diagonal  $\varphi_1 = \varphi_2$ , and have a kink on the diagonal.

We write a spherical distribution  $\Phi_{c,l}^r$  corresponding to a signature (3.4) in the form

$$\langle f, \Phi_{c,l}^r \rangle = \sum \gamma_k \int_{A_k} \Xi_k f(a^{(k)}) \varkappa_k(c, l; a^{(k)}) da^{(k)}, \quad (3.9)$$

where  $\gamma_k$  are given by (1.6) and  $\varkappa_k(c, l; a^{(k)})$  will be defined now.

Let  $k \leq r$ . We consider diagrams  $\mathfrak{S}$  of the form shown on Fig.3. The upper row consists of  $n - 2r$  white circles corresponding to the parameters  $c_1$ ,

$\dots, c_{n-2r}$  and  $2r$  black circles corresponding to the parameters  $l_1, \bar{l}_1, l_2, \bar{l}_2, \dots, l_r, \bar{l}_r$ . The lower row consists of  $n - 2k$  white boxes corresponding to the coordinates  $\varphi_1, \dots, \varphi_{n-2k}$  and  $2k$  black boxes corresponding to the coordinates  $z_1, -\bar{z}_1, z_2, -\bar{z}_2, \dots, z_k, -\bar{z}_k$ . In the upper row, horizontal dotted arcs link elements of pairs  $(l_q, \bar{l}_q)$ ; in the lower row, elements of pairs  $(z_\gamma, -\bar{z}_\gamma)$ . Vertical arcs establish a bijection between circles and a boxes. This bijections are not arbitrary, vertical arcs must belong to one of the following 3 types:

Type 1. An arc connecting a white circle and a white box. We denote such arcs by  $[c_p; \varphi_\alpha]$  according the variables on its ends. The number of such arcs is  $n - 2r$ .

Type 2. Pairs of arcs from linked black circles to white boxes,  $[l_p, \varphi_\alpha], [\bar{l}_p, \varphi_\beta]$ , where  $\alpha < \beta$ . The number of such pairs is  $r - k$ .

Type 3. Pairs of non-intersecting arcs from linked black circles to linked black boxes  $[l_p, z_\gamma], [\bar{l}_p, -z_\gamma]$ . The number of such pairs is  $k$ .

Denote the set of such diagrams by  $\Omega(r, k)$ .

Since  $\mathfrak{S}$  determines a permutation, it has a well-defined sign  $(-1)^{\mathfrak{S}}$ . This number is also a parity of the number of intersections of the vertical arcs, notice that we can do not take to an account arcs of Type 3.

Functions  $\varkappa_k$  determining spherical distributions (see 3.9) are given by formula:

$$\varkappa_k(c, l; a^{(k)}) := 0 \quad \text{for } k > r; \quad (3.10)$$

$$\begin{aligned} \varkappa_k(c, l; a^{(k)}) = & \sum_{\mathfrak{S} \in \Omega(r, k)} \prod_{[c_p, \varphi_\alpha] \in \mathfrak{S}} e^{i c_p \varphi_\alpha} \prod_{[l_q, z_\gamma] \in \mathfrak{S}} \xi(l_q; e^{z_\gamma}) \times \\ & \times \prod_{[l_q; \varphi_\alpha], [\bar{l}_q, \varphi_\beta] \in \mathfrak{S}} D(l_q; e^{i \varphi_\beta}, e^{i \varphi_\gamma}) \quad \text{for } k \leq r, \end{aligned} \quad (3.11)$$

the summation is taken over all diagrams  $\mathfrak{S}$ , the product is taken over all pieces of a given diagram  $\mathfrak{S}$  (see [22], equation (7.6), and the expression  $\varkappa$  at the last row of §7).

REMARK. The expressions  $\varkappa_k(c, l; a^{(k)})$  are invariant with respect to the subgroups  $S_{n-2k}, S_k \subset W_k$  acting on  $A_k$  and and changes a sign under reflections (1.5). Also, they are invariant with respect to permutations of  $c_1, \dots, c_{n-2r}$  and with respect to permutations of  $l_1, \dots, l_r$ . They change a sign under the reflections

$$(\dots, l_{j-1}, l_j, l_{j+1}, \dots) \mapsto (\dots, l_{j-1}, \bar{l}_j, l_{j+1}, \dots). \quad (3.12)$$

**3.4. Functions  $\varkappa(c, l; a^{(k)})$  as eigenfunctions of symmetric differential operators.** Notice, that the functions  $\xi, D$  can be written as

$$\xi(l; z) = e^{z l + \bar{z} \bar{l}} - e^{\bar{z} l + z \bar{l}}, \quad (3.13)$$

$$D(l; e^{i \varphi_1}, e^{i \varphi_2}) = \frac{2}{e^{2 \pi i l} - 1} e^{i \varphi_1 l + i \varphi_2 \bar{l}} - \frac{2}{e^{2 \pi i \bar{l}} - 1} e^{i \varphi_1 \bar{l} + i \varphi_2 l}, \quad (3.14)$$

in the second expression we choose  $\varphi_1/2 > \varphi_2/2$ .

Therefore locally the functions  $\varkappa_k(\varphi, z)$  are linear combinations of exponents of linear functions.

Consider a symmetric polynomial  $S(x_1, \dots, x_n)$ . Substituting the first order differential operators (1.11) to  $S(\cdot)$  we come to the identity

$$S(X_1, \dots, X_n) \varkappa_k(c, l; a^{(k)}) = S(\lambda_1, \dots, \lambda_n) \varkappa_k(c, l; a^{(k)}),$$

which holds outside hypersurfaces  $\varphi_p = \varphi_q$ . Moreover, this identity is valid in a distributional sense. Precisely, for any  $f \in C^\infty(G)$

$$\begin{aligned} \sum_{k=0}^{[n/2]} \gamma_k \int_{A_k} S(-X) \Xi_k f(a^{(k)}) \cdot \varkappa_k(a^{(k)}) da^{(k)} &= \\ &= \sum_{k=0}^{[n/2]} \gamma_k \int_{A_k} \Xi_k f(a^{(k)}) \cdot S(X) \varkappa_k(a^{(k)}) da^{(k)}. \end{aligned}$$

Boundary terms, which appear after integration by parts, cancel due the gluing conditions, see [22], Lemma 6.2. Sano also establishes a more general integration by parts identity (it will be used below). Let  $S(x_1, \dots, x_n)$ ,  $T(x_1, \dots, x_n)$  be homogeneous polynomials, which are both symmetric or both skew-symmetric in  $x_1, \dots, x_n$ . Consider differential operators  $S(X_1, \dots, X_n)$ ,  $T(X_1, \dots, X_n)$ , where  $X_j$  are given by (1.11). Then

$$\begin{aligned} \sum_{k=0}^{[n/2]} \gamma_k \int_{A_k} \Xi_k f(a^{(k)}) \cdot S(X) T(X) \varkappa_k(a^{(k)}) da^{(k)} &= \\ &= \sum_{k=0}^{[n/2]} \gamma_k \int_{A_k} S(-X) \Xi_k f(a^{(k)}) \cdot T(X) \varkappa_k(a^{(k)}) da^{(k)}. \end{aligned} \quad (3.15)$$

**3.5. The Plancherel Theorem.** Consider the distributions  $\Phi_{c,l}^r$  on  $G/H$  given by (3.9). Denote (see formula(1.3))

$$\Delta(c, l) := \Delta(c_1, \dots, c_{n-2r}, l_1, \bar{l}_1, \dots, l_r, \bar{l}_r).$$

Then the following Plancherel formula holds. For any  $f \in C_c^\infty(G/H)$ ,

$$\begin{aligned} \gamma_* f(e) &= \frac{1}{(2\pi)^n} \sum_{r=0}^{[n/2]} \frac{(n-2r)! i^{[n/2]-r} (-1)^r}{([n/2]-r)!} \times \\ &\times \sum_{c_1 > \dots > c_{n-2r}} \sum_{m_1, \dots, m_r} \int_{\lambda_1 > \dots > \lambda_r > 0} \langle f, \Phi_{c,l}^r \rangle \Delta(c, l) d\lambda_1 \dots d\lambda_r. \end{aligned} \quad (3.16)$$

where  $\Phi_{c,l}^r$  are spherical distributions given by (3.9) and  $\gamma_*$  is the same constant as in (1.12). See [22], Theorem 8.7. The formula assumes that summands of the exterior sum  $\sum_r$  are absolutely convergent (as integrals over measures on spaces  $\mathbb{Z}^{n-2r} \times \mathbb{Z}^r \times \mathbb{R}_+^r$ ).

## 4 Calculations

Thus, we must evaluate summands of the exterior sum in (3.16), i.e. we must find a distribution given by

$$\begin{aligned} \gamma_*(2\pi)^n \langle f, \Theta_r \rangle &= \frac{(n-2r)! i^{[n/2]-r} (-1)^r}{([n/2]-r)!} \sum_{c_1 > \dots > c_{n-2r}} \sum_{m_1, \dots, m_r} \times \\ &\times \int_{\lambda_1 > \dots > \lambda_r > 0} \left( \sum_{k=0}^r \gamma_k \int_{A_k} \Xi_k f(a^{(k)}) \cdot \varkappa_k(c, l; a^{(k)}) da^{(k)} \right) \Delta(c, l) d\lambda_1 \dots d\lambda_r. \end{aligned} \quad (4.1)$$

**4.1. Integration by parts.** Let  $l$  be as above,  $l = (m - i\lambda/2)$ . Define functions

$$\xi'(l; e^z) := -e^{im\theta} (e^{i\lambda\theta} + e^{-i\lambda\theta}).$$

If  $m \in 2\mathbb{Z}$ , we set

$$\begin{aligned} D'(l; e^{i\varphi_1}, e^{i\varphi_2}) &:= 2e^{im(\varphi_1 + \varphi_2)/2} \times \\ &\times \frac{\sinh \lambda(|\lfloor (\varphi_1 - \varphi_2)/2 \rfloor| - \pi/2)}{\sinh \pi\lambda/2} \cdot \text{sign}(\lfloor (\varphi_1 - \varphi_2)/2 \rfloor). \end{aligned}$$

If  $m \in 2\mathbb{Z} + 1$ , then

$$\begin{aligned} D'(l; e^{i\varphi_1}, e^{i\varphi_2}) &:= -2e^{i(m-1)(\varphi_1 + \varphi_2)/2} \times \\ &\times e^{i\varphi_2 + i\lceil (\varphi_1 - \varphi_2)/2 \rceil} \frac{\cosh \lambda(\lceil (\varphi_1 - \varphi_2)/2 \rceil - \pi/2)}{\cosh \lambda\pi/2}. \end{aligned} \quad (4.2)$$

A function  $D(l; e^{i\varphi_1}, e^{i\varphi_2})$  has a jump on the diagonal  $e^{i\varphi_1} = e^{i\varphi_2}$  and  $C^\infty$ -smooth outside the diagonal.

Following [22], consider functions  $\varkappa'(c, l; a^{(k)})$  on the union of  $A_k$  given by the formula

$$\varkappa'_k(c, l; a^{(k)}) := 0 \quad \text{for } k > r. \quad (4.3)$$

$$\begin{aligned} \varkappa'_k(c, l; a^{(k)}) &= \\ &= \sum_{\mathfrak{S} \in \Omega(r, k)} (-1)^{\mathfrak{S}} \prod_{[c_p, \varphi_\alpha] \in \mathfrak{S}} e^{ic_p \varphi_\alpha} \prod_{[l_q, z_\delta] \in \mathfrak{S}} \xi'(l_q; e^{z_\delta}) \times \\ &\times \prod_{[l_q; \varphi_\alpha], [\bar{l}_q, \varphi_\beta] \in \mathfrak{S}} D'(l_q; e^{i\varphi_\alpha}, e^{i\varphi_\beta}) \quad \text{for } k \leq r. \end{aligned} \quad (4.4)$$

REMARK. These functions are skew-symmetric with respect to the variables  $\varphi_m$  and invariant with respect to the subgroup  $S_k \times \mathbb{Z}_2^k \subset W_k$ . They are skew-symmetric with respect to the parameters  $c_m$  and invariant with respect to the permutations of  $l_\alpha$  and the reflections (3.12)  $\square$

We have

$$\Delta_k(\partial) \varkappa_k(c, l; a^{(k)}) = \Delta(c, l) \varkappa'_k(c, l; a^{(k)}); \quad (4.5)$$

$$\Delta_k(\partial) \varkappa'_k(c, l; a^{(k)}) = \Delta(c, l) \varkappa_k(c, l; a^{(k)}). \quad (4.6)$$

These identities hold pointwise outside diagonals  $e^{i\varphi_p} = e^{i\varphi_q}$ , this easily follows from (3.13). Moreover, we have an identity:

$$\begin{aligned} \sum_{k \leq r} \gamma_k \int_{A_k} \Xi_k f(a^{(k)}) \cdot \Delta(c, l) \varkappa_k(c, l; a^{(k)}) da^{(k)} &= \\ &= (-1)^{n(n-1)/2} \sum_{k \leq r} \gamma_k \int_{A_k} \Delta_k(\partial) \Xi_k f(a^{(k)}) \cdot \varkappa'_k(c, l; a^{(k)}) da^{(k)} \end{aligned}$$

To obtain this, we apply (3.15) with  $S, T = \Delta$ .

Thus,

$$\gamma_* \cdot (2\pi)^n \langle f, \Pi_r \rangle = \frac{(n-2r)! i^{[n/2]-r} (-1)^r (-1)^{n(n-1)/2}}{([n/2]-r)!} \sum_k \gamma_k U_{r,k},$$

where

$$U_{r,k} = \sum_{c_1 > \dots > c_{n-2r}} \sum_{m_1, \dots, m_r} \int_{\lambda_1 > \dots > \lambda_r > 0} \int_{A_k} h_k(a^{(k)}) \cdot \varkappa'_k(c, l; a^{(k)}) da^{(k)} d\lambda_1 \dots d\lambda_r \quad (4.7)$$

where  $h_k(a^{(k)}) = \Delta_k(\partial) \Xi_k f(a^{(k)})$ .

#### 4.2. Summation of distributions.

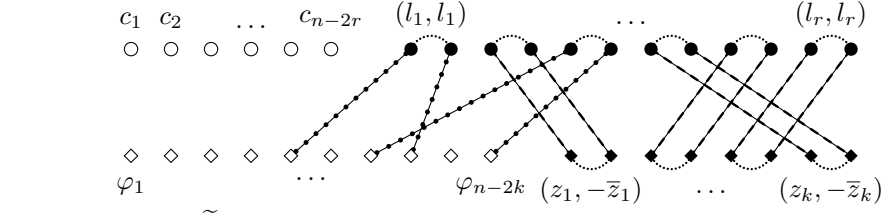
**Lemma 4.1** *Let  $h_k(a^{(k)})$  be skew-symmetric with respect to  $S_{n-2k} \subset W_k$ , symmetric with respect to  $S_k \ltimes \mathbb{Z}_2^k$  and smooth in the domain  $t_1 \geq 0, \dots, t_k \geq 0$ . Then the sum given by (4.7) equals to*

$$\frac{(2\pi)^n (n-2r)! (-2)^{r-k} (-i)^{[n/2]-r}}{(n-2r)!} \left\langle h_k, \Lambda_{n-2k}(\varphi) \prod_{m=1}^k \delta(\theta_m) \delta(t_m) \right\rangle_{A_k}.$$

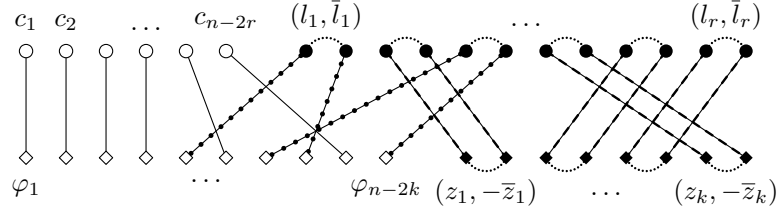
NOTATION FOR A PROOF. For a diagram  $\mathfrak{S} \in \Omega(r, k)$  consider a diagram  $\tilde{\mathfrak{S}}$  obtained by removing vertical arcs of Type 1, i.e., arcs  $[c_p, \varphi_\alpha]$ , see Figure 4.a. Next, we define the diagram  $\mathfrak{S}^\circ \in \Omega(r, k)$  obtained from  $\tilde{\mathfrak{S}}$  by adding a collection of nonintersectig arcs from white circles to free black boxes. Denote by  $\Omega^\circ(r, k)$  the set of diagrams of this form,  $\Omega^\circ(r, k) \subset \Omega(r, k)$ .

For  $\mathfrak{T} \in \Omega^\circ(r, k)$  we denote by  $u_1 < \dots < u_{n-2k}$ , which are connected with white boxes.

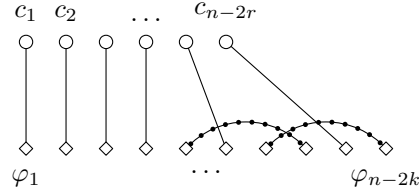




a) The diagram  $\tilde{\mathfrak{S}}$  for  $\mathfrak{S}$  given by Fig.3.



b) The diagram  $\mathfrak{T} = \mathfrak{S}^\circ \in \Omega^\circ(r, k)$



c) The diagram  $\mathfrak{Q} = \mathfrak{T}^\square \in \Omega^\square(r, k)$ .

Figure 4: .

Also for  $\mathfrak{T} \in \Omega^\circ(r, k)$  we define two parities,  $\varepsilon_1(\mathfrak{T})$ ,  $\varepsilon_2(\mathfrak{T})$ ; let  $\varepsilon_1(\mathfrak{T})$  be the parity of number of intersections between arcs of Type 1 and arcs of Type 2. Respectively,  $\varepsilon_2(\mathfrak{T})$  is the parity of the number of intersections of arcs of Type 2,

$$(-1)^{\mathfrak{T}} = (-1)^{\varepsilon_1(\mathfrak{T})}(-1)^{\varepsilon_2(\mathfrak{T})}.$$

Finally, for  $\mathfrak{T} \in \Omega^\circ(r, k)$  we consider the diagram  $\mathfrak{T}^\square$  obtained by the following operation: we forget black circles and black boxes, we forget arcs between black circles and black boxes; we transform any pairs of arcs  $[l_q, \varphi_\alpha]$ ,  $[\bar{l}_q, \varphi_\beta]$  to an arc between  $\varphi_\alpha$ ,  $\varphi_\beta$ . We denote the set of such diagrams by  $\Omega^\square(r, k)$ .

In the same way, for  $Q \in \Omega(r, k)$  we define two numbers  $(-1)^{\varepsilon_1(\mathfrak{Q})}$ ,  $(-1)^{\varepsilon_2(\mathfrak{Q})}$  and the collection  $u_1 < \dots < u_{n-2k}$ .

PROOF. By the symmetry of the expressions  $\varkappa'_k$  in the parameters with

respect to  $S_r \times \mathbb{Z}_2^r$  we can represent this as

$$\begin{aligned}
U_{r,k} &= \\
\frac{1}{r! 2^r} \sum_{c_1 > \dots > c_{n-2r}} \sum_{m_1, \dots, m_r} \int_{\lambda_1, \dots, \lambda_r \in \mathbb{R}} \int_{A_k} h_k(a^{(k)}) \varkappa'_k(c, l; a^{(k)}) da_k d\lambda_1 \dots d\lambda_r = \\
&= \frac{1}{r! 2^r} \sum_{\mathfrak{S} \in \Omega(r, k)} (-1)^{\mathfrak{S}} \sum_{c_1 > \dots > c_{n-2r}} \sum_{m_1, \dots, m_r} \int_{\lambda_1, \dots, \lambda_r \in \mathbb{R}} \int_{A_k} h_k(a^{(k)}) \times \\
&\quad \times \prod_{[c_p, \varphi_\alpha] \in \mathfrak{S}} e^{ic_p \varphi_\alpha} \prod_{[l_q, z_\delta] \in \mathfrak{S}} \xi'(l_q; e^{z_\delta}) \times \\
&\quad \times \prod_{[l_q; \varphi_\alpha], [\bar{l}_q, \varphi_\beta] \in \mathfrak{S}} D'(l_\gamma; e^{i\varphi_\alpha}, e^{i\varphi_\beta}) da_k d\lambda_1 \dots d\lambda_r. \quad (4.8)
\end{aligned}$$

We will use the following identities for distributions. Obviously,

$$\sum_m \int_{\lambda > 0} \xi'(l; z) d\lambda = -(2\pi)^2 \delta(t) \delta(\theta). \quad (4.9)$$

By [22], Lemma 8.4, we have

$$\begin{aligned}
\sum_m \int_{\lambda > 0} D'(l; e^{i\varphi_1}, e^{i\varphi_1}) d\lambda &= 4i \sum_{a_1 > a_2} (e^{ia_1 \varphi_1 + ia_2 \varphi_2} - e^{ia_2 \varphi_1 + ia_1 \varphi_2}) = \\
&= 4i \Lambda_2(e^{i\varphi_1}, e^{i\varphi_2}) = (-8\pi) \cot(\varphi_1/2) \delta(\varphi_1 + \varphi_2). \quad (4.10)
\end{aligned}$$

According [22], we can change order of summation and integration (4.8) in an arbitrary way. We successively integrate with respect to the following groups of variables:

- for each  $[l_q, z_\delta] \in \mathfrak{S}$  we integrate

$$\int d\theta_\delta \int dt_\delta \sum_{m_q} \int d\lambda_q (\dots)$$

applying (4.9);

- for each pair  $[l_q; \varphi_\alpha], [\bar{l}_q, \varphi_\beta] \in \mathfrak{S}$  we integrate

$$\int d\varphi_\alpha \int d\varphi_\beta \sum_{m_q} \int d\lambda_q (\dots)$$

applying (4.9).

We come to

$$U_{r,k} = \frac{(2\pi)^{n-2r}(8\pi^2)^k(-8\pi)^{r-k}}{r! 2^r} \sum_{\mathfrak{T} \in \Omega^\circ(r,k)} \times \\ \times \left\langle h_k, (-1)^{\varepsilon_1(\mathfrak{T})} \Lambda_{n-2r}(e^{i\varphi_{u_1}}, \dots, e^{i\varphi_{u_{n-2r}}}) \times \right. \\ \left. \times (-1)^{\varepsilon_2(\mathfrak{T})} \prod_{[l_q, z_\gamma] \in \mathfrak{T}} \delta(t_\gamma) \delta(\theta_\gamma) \prod_{[l_q; \varphi_\alpha], [\tilde{l}_q, \varphi_\beta] \in \mathfrak{T}} \cot(\varphi_\alpha/2) \delta(\varphi_\alpha + \varphi_\beta) \right\rangle.$$

A summand corresponding to  $\mathfrak{T}$  actually depends on  $\mathfrak{T}^\square$ , and there are  $r!$  different  $\mathfrak{T}^\square$  for a given  $\mathfrak{T}$ . We also apply the expression (2.6) for  $\Lambda$  and get

$$U_{r,k} = \frac{(2\pi)^{n-2r}(8\pi^2)^k(-8\pi)^{r-k} r! (-i)^{[(n-2r)/2]}}{r! 2^r} \times \\ \times \left\langle h_k, \prod_{j=1}^k \delta(t_j) \delta(\theta_j) \times \right. \\ \times \left\{ \sum_{\Omega \in \Omega^\square(r,k)} (-1)^{\varepsilon_1(\Omega)} \sum_{\zeta \in \text{Match}(\{u_1, \dots, u_{n-2r}\})} (-1)^\zeta \prod_{\{\alpha, \beta\} \in \zeta} \cot(\varphi_\alpha/2) (\delta(\varphi_\alpha + \varphi_\beta)) \right. \\ \left. \times (-1)^{\varepsilon_2(\Omega)} \prod_{\{\gamma\} \in \zeta} \delta(\varphi_\gamma) \cdot (-1)^{\varepsilon_2(\Omega)} \prod_{[\beta, \gamma] \in \Omega} \cot(\varphi_\alpha/2) \delta(\varphi_\alpha + \varphi_\beta) \right\} \right\rangle.$$

The sum in the big curly brackets is  $\Lambda_{n-2k}(e^{i\varphi_1}, \dots, e^{i\varphi_{n-2k}})$ . This completes the calculation.  $\square$

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